

Exclusion dynamics and realisation

PDT Foundations R05

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Purpose and dependencies

This paper constructs a complete finite sequence in which each conditional record excludes one candidate and a final label survives. It uses the quantum bridge of R04 and a specified apparatus instrument. The construction supplies exact probabilities, state updates and a pulse implementation. It also exposes a remaining boundary. A conditional exclusion inside a quantum branch does not make that branch actual.

The postulate **E1** is the specific exclusion instrument below. **E2** is its specified pulse or jump realisation when either physical implementation is used. **E3** would be an independent intrinsic trigger or protected exact exclusion law. No E3 law is claimed to have been derived here. E1 and E2 are consistent candidates for Dan to review, not extra universal axioms silently appended to PDT.

An exact finite instrument

Let $\{P_i\}_{i=1}^d$ be nonzero mutually orthogonal projectors summing to the identity. The projectors may have any rank. For an active label set S of size $m \geq 2$, put $Q_S = \sum_{i \in S} P_i$. On states supported in Q_S , define

$$K_j^S = \frac{Q_S - P_j}{\sqrt{m-1}}, j \in S.$$

The record j says that label j has been excluded. The remaining mask is $S \setminus \{j\}$. Outside the declared active sector the operation would need a completion, which plays no role in this theorem.

Theorem R05.1 Complete last survivor instrument. These Kraus operators are complete on the active sector. Starting with $p_i = \text{Tr}(\rho P_i)$, every positive probability history remains nonempty and ends after $d - 1$ exclusions. For any ordered exclusion history h leaving label a ,

$$K_h = \frac{P_a}{\sqrt{(d-1)!}}, \text{Pr}(h) = \frac{p_a}{(d-1)!}.$$

Summing histories with the same survivor gives terminal operation $\rho \mapsto P_a \rho P_a$ and probability p_a .

Proof. Each projector P_i occurs in exactly $m - 1$ of the effects $(K_j^S)^\dagger K_j^S$. Their sum is Q_S . The probability of excluding j is $(1 - p_j)/(m - 1)$, where the p_j are normalised populations in the current sector. Conditional on that event, its population is zero and each other population is divided by $1 - p_j$. A populated singleton is never excluded because that would have probability zero. Each step removes one label, so after $d - 1$ steps one remains. Along an ordered history the commuting projector product is P_a and the denominator is $\sqrt{(d-1)!}$. There are $(d-1)!$ orders excluding the other labels. Summing their operations proves the claim. ■

For $d=3$ and populations $(1/2, 1/3, 1/6)$, the two histories leaving label 1 each have probability $1/4$. Those leaving labels 2 and 3 each have probabilities $1/6$ and $1/12$, respectively. All six histories sum to one and the survivor totals recover the original populations. This worked example is exactly enumerated in the accompanying verification file.

Proposition R05.2 What symmetry fixes. Restrict to rank one sectors and require every active label permutation to be a symmetry of the same apparatus. Among diagonal effects covariant under those permutations and giving zero probability for excluding their own occupied sector, E1's effects are forced. Kraus phases are not forced.

Proof. The stabiliser of label j makes its diagonal effect coefficient equal on all other labels. The coefficient on j is zero. Completeness fixes the common coefficient to $1/(m-1)$. Multiplying the square root of the effect by an outcome dependent unitary leaves the effect unchanged. ■

In a rank one basis, after k unread exclusions an initial off diagonal entry is multiplied by $(d-1-k)/(d-1)$. To see this, count ordered prefixes avoiding both indices. Their number is $(d-2)!/(d-2-k)!$, and each contributes $(d-1-k)!/(d-1)!$. The product gives the stated factor for $k \leq d-2$. At the final step all such entries vanish. No such unread dephasing proves that a particular history is actual.

A bounded pulse implementation

For each $i \in S$ define orthogonal probe components

$$|u_s\rangle = \frac{1}{\sqrt{m}} \sum_{j \in S} |j\rangle, |e_i\rangle = \frac{1}{\sqrt{m-1}} \sum_{j \neq i} |j\rangle.$$

Prepare the probe in $|u_s\rangle$ and adopt

$$H_s = i \hbar \Omega \sum_{i \in S} P_i \otimes (|e_i\rangle\langle i| - |i\rangle\langle e_i|).$$

Proposition R05.3 Pulse and leakage. The Hamiltonian is Hermitian, has norm $\hbar|\Omega|$ on its active blocks, and at pulse angle $\theta_c = \arctan(1/\sqrt{m-1})$ realises E1 upon probe measurement. An angle error δ gives probability $\sin^2 \delta$ for the same label probe record when the system occupied sector i .

Proof. In the plane spanned by $|i\rangle$ and $|e_i\rangle$, the exponential is a real rotation. The amplitude for probe label i is

$$a_m(\theta) = \frac{\cos \theta - \sqrt{m-1} \sin \theta}{\sqrt{m}}.$$

Each other label has amplitude

$$b_m(\theta) = \frac{\sin \theta + \sqrt{m-1} \cos \theta}{\sqrt{m(m-1)}}.$$

At θ_c these are zero and $1/\sqrt{m-1}$. The measured probe matrix element is therefore K_j^S . The rotation generator has eigenvalues $\pm \hbar \Omega$. Substituting $\theta_c + \delta$ gives $a_m = -\sin \delta$. ■

Before the probe is read, the joint state is the coherent isometry $\sum_j K_j^s |\psi\rangle |j\rangle$. Every branch has its own excluded label. Unitary dynamics has not deleted all but one branch of this joint state. If every intermediate probe result is called an actual record, actualisation has already been invoked at those steps. The construction cannot then be used as an assumption free derivation of actualisation at the final step.

A timed conditional process

One possible open system model uses an active mask register and jump operators

$$L_{sj} = \sqrt{\frac{\Gamma}{m-1}} (Q_s - P_j) \otimes |S \setminus \{j\}\rangle \langle S|,$$

with singleton masks absorbing. A fixed jump unravelling is part of E2. On each unresolved mask the total jump rate is Γ , since $\sum_j L_{sj}^\dagger L_{sj} = \Gamma Q_s$. Hence the waiting times are independent exponentials of rate Γ , and their sum has the Erlang law

$$\Pr(T \leq t) = 1 - e^{-\Gamma t} \sum_{k=0}^{d-2} \frac{(\Gamma t)^k}{k!}, \quad E T = \frac{d-1}{\Gamma}.$$

The jump label law is Theorem R05.1. Termination is almost sure with no fixed finite upper bound on all sample paths. The rate, reservoir limit and chosen unravelling have been supplied. A master equation alone does not uniquely select an actual record unravelling. Any energy released to a bath must also appear in a total physical energy account.

Generic noise defeats exact deletion

Theorem R05.4 No open neighbourhood of exact zero exclusion. For every exact finite exclusion effect there are arbitrarily close complete effects with no zero eigenvalue on the active sector.

Proof. Replace the effects by

$$E_j^{(\eta)} = (1 - \eta) E_j + \frac{\eta}{m} Q_s, \quad 0 < \eta < 1.$$

They remain complete and tend to the exact effects as $\eta \rightarrow 0$. Every eigenvalue on the active sector is at least η/m . In the diagonal implementation their positive square roots have strictly positive entries. Any finite product therefore retains a nonzero population for every initially populated label. Deleting a label from the classical mask would no longer describe a zero produced by these physical updates. ■

This obstruction is about unrestricted small full rank errors. It does not rule out an explicitly protected error class or a separately specified finite tolerance record law. Such a law would be E3 and would need its own consequences and tests. Calling a small amplitude exactly impossible is not a mathematical repair.

For practical memory there is a weaker positive result. If an accessible record has joint state $\sum_r c_r |r\rangle |e_r\rangle$ and $|\langle e_s, e_r \rangle| \leq e^{-B}$ for distinct histories, its reduced state is within trace distance

$$\frac{1}{2} e^{-B} \sum_{r \neq s} |c_r c_s| \leq \frac{R-1}{2} e^{-B}$$

of complete dephasing, where R is the number of histories. The triangle inequality for the off diagonal matrix units proves the first bound, and Cauchy Schwarz proves the second. Subsequent accessible channels cannot increase this distance. This is a tolerance dependent operational irreversibility bound. It does not select a realised branch or forbid reversal when the environment becomes controllable.

Coherence burdens that define valid channels

A coherence burden is constrained if it is to produce a legitimate quantum channel. Let ψ be a continuous real even function on the phase circle with $\psi(0)=0$. For phase labels θ_a , consider the dephasing multiplier $K_{ab}(s) = e^{-s\psi(\theta_a - \theta_b)}$. A positive semidefinite matrix K with diagonal one defines a completely positive trace preserving Schur channel $\rho_{ab} \mapsto K_{ab} \rho_{ab}$. Factor K as a Gram matrix to obtain diagonal Kraus operators and verify their completeness.

Theorem R05.5 Admissible circle dephasing burdens. The multiplier is positive semidefinite for every finite phase list and every $s \geq 0$ exactly when

$$\psi(\theta) = \sum_{n \geq 1} c_n (1 - \cos n\theta), c_n \geq 0, \sum_n c_n < \infty.$$

Proof. Positivity at every s , differentiated from the right at zero against coefficients whose sum is zero, makes ψ conditionally negative. Approximating integrals by finite sums extends this inequality to continuous zero mean test functions. Applying it to each nonzero Fourier mode gives $\hat{\psi}(n) \leq 0$. Since ψ is real and even, put $c_n = -2\hat{\psi}(n) \geq 0$. Fejér convergence at zero gives $\hat{\psi}(0) = \sum_n c_n$. Indeed the weighted partial sums of these nonnegative coefficients increase to the full sum. This is finite. The resulting absolutely convergent cosine series has the Fourier coefficients of ψ , so it equals ψ .

Conversely, for one coefficient the function $\exp[-s c_n (1 - \cos n\theta)]$ is the characteristic function of a compound Poisson integer variable with symmetric jumps $+n$ and $-n$. Characteristic functions give positive semidefinite difference kernels. Independent sums multiply these kernels, and the summable series gives a uniform limit preserving positivity. ■

This is a circle form of standard positive kernel mathematics [B18], retained from the earlier PDT coherence analysis. It restricts permissible burdens but leaves every coefficient unselected. In particular it does not force the first harmonic alone. For finite s every displayed multiplier is nonzero. Valid dephasing and literal finite candidate exclusion therefore remain different claims. A physical phase identification, exposure parameter and interaction selecting the coefficients still belong to Q4.

Extension to a supplied instrument

For a finite instrument from R04, the map

$$V|\psi\rangle = \sum_{a,\alpha} L_{a\alpha}|\psi\rangle \otimes |a\rangle \otimes |\alpha\rangle$$

is an isometry because $V^\dagger V = I$. It can be extended to a unitary on a sufficiently enlarged finite space. Apply Theorem R05.1 to the orthogonal outcome blocks $I \otimes |a\rangle\langle a| \otimes I$. Tracing the auxiliary registers after the terminal outcome gives exactly the original operation I_a .

Thus a last survivor representation exists for every supplied finite instrument. Its universality also shows its limitation. By itself it selects no particular instrument, frequency law or distinctive experiment. A6 interprets a physical survivor, while E3 and the physical choice of Q4 remain open if an intrinsic PDT record mechanism is claimed.

Sources

[S01] *PDT Foundations and Physical Closure*, 12 September 2026, Sections 8 and 9. The exclusion instrument, pulse law, timed process, noise obstruction and trace distance bound are reconstructed here from this source.

[S04] *PDT Conditional Closure of Quantum Mechanics*, 13 September 2026.

[B05] S. Bandyopadhyay and colleagues, “Conclusive Exclusion of Quantum States”, *Physical Review A* 89, 022336 (2014). Author manuscript. Established state exclusion context, not a proof of PDT actualisation.

[B18] I. J. Schoenberg, “Metric Spaces and Positive Definite Functions”, *Transactions of the American Mathematical Society* 44, 522–536 (1938). Original paper scan.