

Phase transport and complex structure

PDT Foundations R02

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Purpose and dependencies

This paper uses A0 through A4 from R01. It establishes what relational phase can represent before introducing physical quantum states. A circle valued transport can carry nontrivial loop information, and its minimal faithful continuous real linear carrier is a plane. Neither fact supplies the laws of quantum mechanics by itself.

Vertex phases and loop phases

Suppose a vertex assignment $v: X \rightarrow T$ is used to define $u_{AB} = v(B)v(A)^{-1}$. On every closed chain the factors telescope,

$$u_{A_{m-1}A_0} \cdots u_{A_1A_2} u_{A_0A_1} = 1.$$

This exact construction cannot supply nontrivial loop phase. It is a special case of A2 and must not be treated as the general case.

Instead assign independent phases to the oriented edges of a graph, give each reversed edge the reciprocal phase, and multiply along paths. Reduced paths form the free path groupoid. Cancellation of an edge with its reverse proves that the assignment extends to a functor. Under a change of vertex frame v_x , an edge from x to y transforms as

$$u_{xy} \mapsto v_y u_{xy} v_x^{-1}.$$

Closed path products are unchanged because the residual conjugation is trivial in $U(1)$. This is the usual discrete connection construction. Calling frame changes physically unobservable is a gauge identification adopted in a physical model, not a consequence of naming the variables.

For a triangle, take $u_{12} = u_{23} = 1$ and $u_{31} = e^{i\theta}$. Its loop product is $e^{i\theta}$, which is nontrivial when θ is not a multiple of 2π . This example directly resolves the incompatibility between pure vertex phase differences and the loop dependent orientation used in R07.

Proposition R02.1 The phase quotient. Identify parallel arrows $g, h: x \rightarrow y$ when $\phi(g) = \phi(h)$. This equivalence defines a quotient groupoid, and its induced phase functor is injective on each hom set.

Proof. Replacing either factor in a composable pair by an equivalent arrow does not change the product phase. Reversing equivalent arrows preserves equality of phase. Composition and inverses therefore descend. In each quotient hom set, equality of phase is exactly equality of equivalence classes. ■

The quotient deliberately discards other transport information. It does not prove that the full relational structure is only a phase structure. A commutator in any isotropy group has phase one, so noncommuting information can lie in the kernel of the phase functor.

Minimal real representation

Theorem R02.2 The phase plane. The smallest dimension of a faithful continuous real linear representation of $U(1)$ is two. After choosing an invariant inner product and an angular orientation, the action has the form

$$T_\alpha = \cos \alpha I + \sin \alpha J, J^2 = -I.$$

Proof. A continuous image of the connected compact group $U(1)$ in $GL(1, \mathbb{R})$ lies in the positive real numbers. The only compact subgroup there is the identity, so dimension one cannot be faithful. Ordinary plane rotations supply a faithful representation in dimension two. Average any inner product over the circle to obtain an invariant positive inner product. In that metric the connected image lies in $SO(2)$. A continuous circle homomorphism into $SO(2)$ has integer winding number k . Faithfulness requires $|k| = 1$. Choose the angular orientation to make $k = 1$. The quarter turn J then satisfies $J^2 = -I$, and the stated formula follows. ■

Multiplication by i can consequently be represented by J . This is a mathematical complex structure on the selected carrier. It does not determine tensor composition, physical superposition, the Born rule, a Hamiltonian or the dimension of space. In particular the representation class in A4 already assumed real linearity.

A Hilbert representation and its limitation

For each object x , take the Hilbert space with orthonormal basis indexed by arrows ending at x ,

$$H_x = \ell^2(\gamma : t(\gamma) = x), U(g)|\gamma\rangle = \phi(g)|g \circ \gamma\rangle.$$

Proposition R02.3 Regular transport. Each $U(g)$ is unitary between the corresponding fibres, and $U(h)U(g) = U(h \circ g)$.

Proof. Composition by an invertible arrow bijects the two basis sets. Its scalar multiplier has modulus one. It therefore preserves the inner product and is onto. The functor law for ϕ proves the composition identity. ■

This construction is valid even for an uncountable discrete basis set, with the usual square summable Hilbert space convention. It does not establish strong continuity in a topology on the arrow set. For example continuously varying an arrow can move its basis vector to an orthogonal vector. A physical continuous time action cannot be inferred merely from the existence of these unitary maps.

There is a second useful limitation. Set $W_x|\gamma\rangle = \phi(\gamma)|\gamma\rangle$, and let $L(g)$ denote unweighted left composition. Then

$$U(g) = W_y L(g) W_x^{-1}.$$

Thus this particular phase twist is removable by a Hilbert basis change. The construction does not select a unique physical phase dynamics.

Proposition R02.4 A finite sector obstruction. If one regular bisection acts by a basis permutation up to the preceding removable twist, every finite dimensional invariant unitary sector has eigenvalues that are roots of unity.

Proof. Decompose the permutation into finite cycles and infinite orbits. On an infinite orbit an eigenvector would have constant coefficient modulus and hence could not be nonzero and square summable. On a cycle of length m , an eigenvalue satisfies $\lambda^m = 1$. Each eigenvector of a finite invariant unitary sector lifts to an eigenvector of the full permutation. Its eigenvalue must therefore be a root of unity. A common multiple of the finitely many orders makes the restricted operation periodic. ■

Coherent linear combinations of represented transports may escape this restriction, but treating them as physical operations is an additional bridge. R04 states an operational bridge rather than concealing it in the word representation.

Refinement requires compatibility

A presentation enlarged by new generators and relations admits an extension of its phase functor precisely when phases can be assigned to the new generators satisfying all the new relations and the previous assignments. When such assignments are fixed, the universal property of a presentation gives a unique functor. It does not guarantee an assignment exists or that the assignment is unique.

For a concrete obstruction, map $\mathbb{Z}$ into $\mathbb{Z}$ by $n \mapsto 2n$ and let the source generator have value -1 in the two element group. An extension would require a square root of -1 in that target, which does not exist. With target $U(1)$ square roots do exist, but they need not be unique. Refinement compatibility and uniqueness are different obligations.

A burden cannot be an additive groupoid cost

Proposition R02.5. A nonnegative real function B on every arrow of a groupoid, additive under every compatible composition, is identically zero.

Proof. Additivity first gives $B(1_x) = 0$. Then $0 = B(g^{-1}g) = B(g^{-1}) + B(g)$. Both terms are nonnegative, so both vanish. ■

An accumulating coherence burden must therefore belong to a directed process, a state functional or an environmental interaction. It cannot be a strictly positive additive cost on all invertible comparisons. R03 uses directed episodes and R05 supplies a conditional detector process.

Sources and onward link

[S01] *PDT Foundations and Physical Closure*, 12 September 2026, Section 3.

[S02] *PDT Axioms and Conditional Dynamical Closure*, 13 September 2026, Sections 3, 4 and 8. The regular bisection obstruction and distinction between representation and physical operation are retained from this source.

[B01] Tom Leinster, *Basic Category Theory*, [arXiv 1612.09375](https://arxiv.org/abs/1612.09375).

R03 next treats what the structural carrier leaves unsettled about selection, retained records and the last survivor.