

Quantum states probability and dynamics

PDT Foundations R04

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Purpose and result

R02 supplies complex mathematical carriers. This paper states the extra operational assumptions that make such a carrier a quantum state space. It then derives the density operator probability law and unitary reversible evolution in finite dimension. These are established mathematical results used within a conditional PDT completion. They are not consequences of bare plurality or A6 alone.

The physical inputs are named so that Dan can assess them individually. **Q1** identifies a finite complex Hilbert space and its full effect space with physically meaningful preparations and tests, using probabilities independent of the effect's measurement context. **Q2** requires mixture consistency, normalisation and positivity with every untouched quantum ancilla, and gives a closed reversible operation an inverse with the same properties on the same sector. **Q3** supplies a continuous coherent time group when a continuous time description is used. **Q4** supplies the apparatus instrument and preparation. Composite system identification, including the ancillas invoked in Q2, is examined in R06. These are a joint operational package, not deductions from one another.

From effects to density operators

For one preparation, let $g(E)$ be its probability for an effect $0 \leq E \leq I$. Q1 says the same effect has the same probability in every finite positive operator valued measurement containing it. For every list with $\sum_j E_j = I$, require $g(E_j) \geq 0$ and $\sum_j g(E_j) = 1$. The full effect space matters. A restricted list of projective measurements would be a different premise.

Theorem R04.1 The finite effect probability theorem. Under Q1 there is a unique positive matrix ρ with trace one such that

$$g(E) = \text{Tr}(\rho E).$$

Proof. Comparing the measurements $(E, F, I - E - F)$ and $(E + F, I - E - F)$ proves additivity when $E + F \leq I$. It also gives $g(0) = 0$. If $E \leq F$, then $g(F) = g(E) + g(F - E) \geq g(E)$, so the function is monotone. Repeated addition gives rational homogeneity on allowed multiples. Approximate a real multiplier by increasing and decreasing rational multipliers and use monotonicity to obtain real homogeneity.

For any positive matrix A , choose $t > 0$ with $t \geq \|A\|$ and set $G(A) = t g(A/t)$. Homogeneity makes the choice of t irrelevant. Scaling two positive matrices into one common effect range proves additivity of G . Extend to Hermitian matrices by differences of positive matrices. If $A - B = C - D$, then $A + D = B + C$, so the extension is well defined. It is a real linear positive functional on the finite dimensional real space of Hermitian matrices.

The trace pairing gives a unique Hermitian matrix ρ representing this functional. Positivity on rank one projectors makes ρ positive. Its value on I gives trace one. Restriction back to effects proves the result. ■

This is the finite dimensional form of the Busch effect theorem [B02]. Q1 already encodes a strong physical relation between distinct measurement procedures. The theorem does not derive that relation from relational phase. Nor does it say that a hidden actualisation variable must assign context independent zero or one values to every effect.

What reversible operations imply

Q2 identifies the full finite quantum preparation space and all the required ancillas. Mixture consistency extends an operation affinely from density matrices and then linearly to matrices. Normalisation gives trace preservation. Positivity under every ancilla extension gives complete positivity. In finite dimension it therefore has Kraus operators,

$$E(\rho) = \sum_i A_i \rho A_i^\dagger, \sum_i A_i^\dagger A_i = I.$$

Theorem R04.2 Reversible channels on one sector. If E and its inverse are completely positive and trace preserving on the same finite matrix space, then $E(\rho) = U \rho U^\dagger$ for a unitary U .

Proof. Let B_j be Kraus operators for the inverse. The identity channel has a rank one Choi matrix. Every vectorised product $B_j A_i$ in a Kraus representation of that channel must therefore be proportional to the identity, so $B_j A_i = c_{ji} I$. Some coefficient is nonzero. Fix such a j , for which B_j is invertible. Every $A_i = c_{ji} B_j^{-1}$, so all nonzero Kraus operators are proportional to one matrix. Trace preservation makes its suitably normalised representative unitary. ■

The equal input and output sector and the physical inverse are essential [B03]. A2 gives inverse comparisons, which are different objects. A5 gives retained history, which is not reversible physical dynamics.

Continuous evolution and its free generator

Adopt Q3 as a continuous amplitude group $U(t)$ on the chosen sector, with $U(t+s) = U(t)U(s)$ and $U(0) = I$. The parameter acquires measured units only through a clock identification in R08.

Proposition R04.3 Finite continuous generator. There is a Hermitian H such that $U(t) = \exp(-i t H / \hbar)$.

Proof. For sufficiently small $a > 0$, the matrix $C = \int_0^a U(v) dv$ is invertible because $C/a \rightarrow I$. The

group law gives $U(t)C = \int_t^{t+a} U(v) dv$. The right side is differentiable by continuity, hence so is U .

Differentiating the group law gives $U'(t) = U(t)A$ with $A = U'(0)$. Differentiating unitarity gives $A^\dagger = -A$. Set $H = i \hbar A$ and solve the matrix differential equation. ■

This finite proof avoids imposing differentiability separately. In an infinite dimensional physical extension, strong continuity gives the standard self-adjoint generator theorem, but its domain and

the physical identification still need to be specified. The present finite result does not establish an interacting continuum field theory.

One observed time step does not determine H . If $U(\delta t) = \sum_j e^{-i\theta_j} P_j$, then all

$$H_n = \frac{\hbar}{\delta t} \sum_j (\theta_j + 2\pi n_j) P_j, n_j \in \mathbb{Z},$$

give that step. Relative integer shifts change physical energy differences. Even an already calibrated $\hbar$ does not remove this freedom. A specified Hamiltonian or additional continuous dynamics is needed.

Linearity is not a consequence of phase language

On a full torus of independent coordinates $z_1, \dots, z_r$, a relational coordinate substitution can be a character product,

$$F_M(z)_j = \prod_k z_k^{M_{jk}}, M_{jk} \in \mathbb{Z}.$$

Proposition R04.4. Such a map is the restriction of a homogeneous real linear map exactly when each row of M is $+e_k$ or $-e_k$. It is complex linear exactly when each row is $+e_k$.

Proof. A real linear component on $\mathbb{C}^r$ is a sum of terms $a_k z_k + b_k \bar{z}_k$. Its torus Fourier support lies only at $+e_k$ and $-e_k$. A character product has one Fourier index, namely the row of M . Uniqueness of Fourier coefficients proves necessity. Coordinate selection or conjugation proves sufficiency. ■

For example $(z_1, z_2) \mapsto (z_1^2 \bar{z}_2, z_1)$ is invertible, preserves Haar measure and respects a common phase rotation. It has no linear extension. Likewise $\psi_a \mapsto e^{i\lambda|\psi_a|^2} \psi_a$ preserves norm and is reversible but is nonlinear for nonzero λ . Additional operational premises are doing real work in Theorem R04.2.

Measurements require an instrument

Q4 supplies operators $L_{a\alpha}$ with $\sum_{a,\alpha} L_{a\alpha}^\dagger L_{a\alpha} = I$. The probability and unnormalised output for record a are

$$p_a = \text{Tr}(\rho E_a), E_a = \sum_{\alpha} L_{a\alpha}^\dagger L_{a\alpha},$$

$$I_a(\rho) = \sum_{\alpha} L_{a\alpha} \rho L_{a\alpha}^\dagger.$$

Two instruments can have the same effects but different retained states. For example replacing $\sqrt{E_a}$ by $V_a \sqrt{E_a}$, with outcome dependent unitaries, preserves all single measurement probabilities but changes later probabilities. A Born frequency theorem cannot stand in for a measurement update law.

With a supplied instrument, appending an orthogonal classical record gives a completely positive history update. Repeated composition gives consistent probabilities for every finite record sequence. A6 may interpret a physically specified singleton event as actual, but it does not select these Kraus

operators. R05 gives an explicit last survivor implementation of an already supplied finite instrument and states exactly where actualisation remains an input.

Closure status and sources

The probability and reversible operation conclusions are proved within Q1 and Q2. A continuous group gives a generator under Q3. A chosen instrument closes finite operational predictions under Q4. Selection of the physical Hilbert space, effect interpretation, Hamiltonian, instrument and reservoir preparation from A0 through A6 remains open.

[S01] *PDT Foundations and Physical Closure*, 12 September 2026, Sections 3, 4 and 14.

[S02] *PDT Axioms and Conditional Dynamical Closure*, 13 September 2026, Sections 3 through 5 and Appendix A.

[S04] *PDT Conditional Closure of Quantum Mechanics*, 13 September 2026. The operational bridge is retained as conditional.

[B02] P. Busch, “Quantum States and Generalized Observables”, *Physical Review Letters* 91, 120403 (2003). Author manuscript.

[B03] A. Nayak and P. Sen, “Invertible Quantum Operations and Perfect Encryption of Quantum States”, *Quantum Information and Computation* 7, 103–110 (2007). Author manuscript.

[B04] J. Watrous, *The Theory of Quantum Information*, 2018. Author manuscript. Finite channels, Kraus representations and the Choi construction.