

Spatial carriers metric and orientation

PDT Foundations R07

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Review draft 1.0 • 24 September 2026

Purpose and dependencies

This paper uses the edge connection construction of R02 to build an oriented three dimensional internal carrier from a chosen four vertex seed. The construction is rigorous on its nondegenerate domain. Selecting that seed and identifying its carrier with physical displacement are further premises. Those distinctions are necessary if emergence is to mean more than assigning a familiar name to a mathematical space.

The local choices are **S1**, a complete four vertex comparison seed with independent edge phase transport, and **S2**, interpretation of its contrast carrier as physical displacement when that interpretation is claimed. **S3** is a specified dynamical barrier if orientation persistence is required. None is part of A0 through A6.

What rank matching establishes

For a complete graph K_N , the vertex contrast space has dimension $N - 1$. Its cycle rank is $|E| - |V| + 1 = (N - 1)(N - 2) / 2$. If one additionally equates the two ranks and requires $N > 1$, then $N = 4$ and both ranks equal three.

This calculation is exact but does not select its own rank equality. The free path groupoid construction of R02 works on every K_N , and R01's histories and distinct keys supply A5 and A6 for any finite $N \geq 2$. The primitive package therefore admits $N = 3$ and $N = 5$ countermodels. Subdividing an edge also increases the contrast rank while preserving cycle rank. Rank matching is not invariant under arbitrary refinement.

The orientation obstruction

Let $V = \left\{ x \in \mathbb{R}^4 : \sum_i x_i = 0 \right\}$ with the permutation action of S_4 .

Proposition R07.1. There is no S_4 equivariant isomorphism $V \cong \Lambda^2 V$. There is a natural isomorphism $\Lambda^2 V \cong V^* \otimes \det V$.

Proof. A transposition has eigenvalues $1, 1, -1$ on V , giving trace one. Its induced eigenvalues on $\Lambda^2 V$ are $1, -1, -1$, giving trace minus one. Isomorphic representations must have equal traces. The map sending $u \wedge v$ to the functional $w \mapsto u \wedge v \wedge w$ gives the stated natural isomorphism in dimension three. ■

A metric identifies V^* with V , while an orientation and volume normalisation trivialise the determinant factor. These are different pieces of information. A claim that rank equality alone provides them would fail this test.

A phase supplied frame

Choose tree edges 12, 13, 14 with unit phase by a vertex gauge choice. Let the remaining edge phases 23, 24, 34 have angles a, b, c . The four unoriented triangular loop scores are

$$f = (\cos(a + c - b), \cos c, \cos b, \cos a).$$

These scores use genuine loop phases, not pure vertex differences. Let $P = I - 11^T/4$, and form the linear map $F_q: \mathbb{R}^3 \rightarrow V$ whose columns are $Pf, P(f^2), P(f^3)$, with powers taken componentwise. The parameter is $q = (a, b, c)$. Set

$$\Delta(q) = \prod_{i < j} (f_j - f_i).$$

Theorem R07.2 Nondegenerate frame and metric. If all four scores are distinct, F_q is an isomorphism. Its ordered columns supply an orientation, and

$$g_q(v, w) = \langle F_q^{-1}v, F_q^{-1}w \rangle$$

is a positive inner product. The construction is gauge invariant and natural under relabelling the vertices.

Proof. The Vandermonde determinant of columns $(1, f, f^2, f^3)$ is Δ . Subtracting column means leaves the determinant unchanged. The first column is orthogonal to the three projected columns and has norm two. Hence the three dimensional volume of F_q is $|\Delta|/2$. It is nonzero exactly when the scores are distinct. The pullback definition of g_q is then positive. Closed loop cosines are gauge invariant. A vertex relabelling permutes the opposite faces, and P commutes with that permutation, proving naturality. ■

The domain is nonempty. For $(a, b, c) = (\pi/6, \pi/4, \pi/2)$ the scores are $(\cos(5\pi/12), 0, \cos(\pi/4), \cos(\pi/6))$, all distinct. A simultaneous inversion of all edge phases leaves the cosines unchanged. The construction does not select a unique physical handedness from reflection symmetric primitives.

There are useful quantitative bounds. Since $|f_i| \leq 1$, write $A = \sum_i f_i^2 \leq 4$. The squared Frobenius norm of the frame obeys

$$\|F_q\|_F^2 \leq 3A - A^2/4 \leq 8.$$

If its singular values are $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$, then $\sigma_1^2 \leq 8$, $\sigma_1 \sigma_2 \leq 4$ and their product is $|\Delta|/2$. Therefore

$$\frac{1}{8} \|v\|^2 \leq g_q(v, v) \leq \frac{64}{\Delta^2} \|v\|^2.$$

The upper bound diverges as the frame degenerates. A bounded phase assignment alone does not control the physical metric near a score collision.

Conditional persistence of orientation

For the chosen cycle basis the least squared edge angle norm at a fixed cycle angle has matrix

$$M = \frac{1}{4} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Indeed the cycle rows $(1, -1, 0, 1, 0, 0)$, $(1, 0, -1, 0, 1, 0)$ and $(0, 1, -1, 0, 0, 1)$ form a matrix A_c . Constrained least squares gives $M = (A_c A_c^T)^{-1}$. This is a local angular lift calculation, not a selection of a physical kinetic energy.

Set $D = \Delta^2 / 4096$. Because each of its six gap factors has magnitude at most two, $0 < D \leq 1$ on a nondegenerate chamber. Adopt S3 as the Hamiltonian

$$H(q, p) = \frac{1}{2I} p^T M^{-1} p - \kappa \log D(q), I > 0, \kappa > 0.$$

Theorem R07.3 Finite energy chamber persistence. A trajectory starting in a chamber with finite energy E exists for all values of the Hamiltonian evolution parameter and never crosses $\Delta = 0$. Every score gap has magnitude at least $2e^{-E/(2\kappa)}$.

Proof. Positivity of the kinetic energy and conservation of H give $D \geq e^{-E/\kappa}$. Bounding the other five score gaps by two gives $D \leq (f_i - f_j)^2 / 4$, establishing the gap estimate. The phase torus is compact, the energy bounds momentum, and the lower bound on D keeps the trajectory away from the singular boundary. The smooth Hamiltonian vector field therefore continues for all parameter values on this compact energy region. A continuous nonzero Δ cannot change sign. ■

The barrier is an adopted dynamical choice. Smooth finite potentials generally allow crossing with a transverse initial velocity. Other divergent barriers also work, so persistence does not uniquely select the logarithm or its coefficient.

Transport and physical interpretation

Frame comparison $T_{ba} = F_b F_a^{-1}$ preserves the constructed metric and orientation and satisfies $T_{cb} T_{ba} = T_{ca}$. These particular frame transports have trivial closed carrier holonomy. This does not make the independent phase edge connection flat. They are distinct mathematical structures and must not be identified silently.

For independently assigned orientation signs on edges of a connected stage graph, consistent vertex orientations exist precisely when every cycle has positive sign product. Choose an orientation at one root and propagate along paths. Cycle positivity is exactly path independence. There are then two global choices. Thus global orientation needs a gluing condition as well as local frames.

S2 still has to identify these carrier vectors with measured displacements, relate adjoining seeds, control refinement and supply a common physical metric. A positive inner product on one contrast space supplies neither a continuum nor a Lorentzian spacetime. [R08](#) and [R10](#) state those later interfaces explicitly.

Sources

[S01] *PDT Foundations and Physical Closure*, 12 September 2026, Section 13. The frame, bounds, barrier theorem and refinement limits are retained from the audited construction.

[S05] *Phase Differential Theory and the Structural Basis of a Quantum Gravity Bridge*, strengthened edition, 13 September 2026. Internal source file
PDT_Foundations_QM_GR_Bridge_Strengthened_2026-09-13.docx.